Solutions
Math 220
HW # 6
November 12, 2018

Exercise 1. *Prove that for all integers $n \geq 1$,*

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof.

Base Case $(n = 1)$

$$1^2 = 1$$

and

$$\frac{1(1+1)(2+1)}{6} = 1$$

so the base case is true.

Ind. Hyp. Assume for some $k \geq 1$

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Ind. Step We wish to prove

$$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Observe

$$\begin{aligned}
1^2 + 2^2 + \cdots + (k+1)^2 &= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k+3)(k+2)}{6}
\end{aligned}$$

□

Exercise 2. *Prove that for all integers $n \geq 1$,*

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n+1)(n+2)}=\frac{n}{2n+4}.$$

(Please do this using induction. While it would be valid to just take the result from class and subtract $\frac{1}{2}$, do not do this.)

Proof.

Base Case $(n = 1)$

$$\frac{1}{2\cdot 3}=\frac{1}{6}$$

and

$$\frac{n}{2n+4}=\frac{1}{2+4}=\frac{1}{6}.$$

So the formula is valid for the base.

Ind. Hyp. Assume for some $k \geq 1$,

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k}{2k+4}.$$

Ind. Step We wish to prove

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(k+2)(k+3)}=\frac{k+1}{2k+6}.$$

Observe,

$$\begin{aligned}
\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(k+2)(k+3)} &= \frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(k+1)(k+2)}+\frac{1}{(k+2)(k+3)} \\
&= \frac{k}{2k+4}+\frac{1}{(k+2)(k+3)} \\
&= \frac{k(k+3)}{2(k+2)(k+3)}+\frac{2}{2(k+2)(k+3)} \\
&= \frac{k^2+3k+2}{2(k+2)(k+3)} \\
&= \frac{(k+2)(k+1)}{2(k+2)(k+3)} \\
&= \frac{k+1}{2k+6}
\end{aligned}$$

□

Exercise 3. *Prove that for integers $n \geq 1$,*

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1.$$

Proof.

<u>Base Case</u> $(n = 1)$

$$1 \cdot 1! = 1 \cdot 1 = 1$$

and

$$(1+1)! - 1 = 2 - 1 = 1$$

so the base case is true.

<u>Ind. Hyp.</u> Assume for some $k \geq 1$,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1.$$

<u>Ind. Step</u> We wish to prove

$$1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1) \cdot (k+1)! = (k+2)! - 1.$$

Observe

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1) \cdot (k+1)! &= 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! \\ &= (k+1)! - 1 + (k+1) \cdot (k+1)! = (k+1)!(1 + (k+1)) - 1 \\ &= (k+1)!(k+2) - 1 = (k+2)! - 1 \end{aligned}$$

□

Exercise 4. *Prove that for all integers $n \geq 0$, $3|(2^{2n} - 1)$.*

Proof.

<u>Base Case</u> $(n = 0)$

$$2^{2(0)} - 1 = 1 - 1 = 0$$

which is divisible by 3, so the base case is verified.

<u>Ind. Hyp.</u> Assume for some $k \geq 0$ that

$$3|(2^{2k} - 1),$$

which is equivalent to saying $2^{2k} - 1 = 3a$, or $2^{2k} = 3a + 1$, for some $a \in \mathbb{Z}$.

<u>Ind. Step</u> We want to prove

$$3|(2^{2k+2} - 1).$$

Observe

$$2^{2k+2} - 1 = 2^2 \cdot 2^{2k} - 1 = 4(3a+1) - 1 = 4 \cdot 3a + 3 = 3(4a+1)$$

so we see that $2^{2k+2} - 1$ is divisible by 3.

□

Exercise 5. *Observe the pattern*

$$\begin{aligned} \left(1-\frac{1}{2}\right) &= \frac{1}{2} \\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) &= \frac{1}{3} \\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) &= \frac{1}{4} \end{aligned}$$

Conjecture a general formula for integers $n \geq 2$, and use induction to prove that formula.

Proof. It appears that the general formula will be, for $n \geq 2$:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{4}\right) = \frac{1}{n}$$

Base Case ($n = 2$)

$$\left(1-\frac{1}{2}\right) = \frac{1}{2}$$

which fits the formula.

Ind. Hyp. Assume for some $k \geq 1$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right) = \frac{1}{k}.$$

Ind. Step

$$\begin{aligned} \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right) &= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right) \\ &= \frac{1}{k}\left(1-\frac{1}{k+1}\right) = \frac{1}{k}\frac{k}{k+1} \\ &= \frac{1}{k+1} \end{aligned}$$

Therefore the formula is verified by induction.

□

Exercise 6. *Prove that for all integers $n \geq 10$, $2^n > n^3$.*

Proof.

Base Case ($n = 10$) $2^10 = 1024$ and $10^3 = 1000$, so $2^10 > 10^3$. This verifies the base case.

Ind. Hyp. Assume for some $k \geq 10$ that $2^k > k^3$.

Ind. Step Multiplying both sides of the induction hypothesis by 2 gives

$$2^{k+1} > 2k^3.$$

Recall that $(k+1)^3 = k^3 + 3k^2 + 3k + 1$Then, since $k \geq 10$, we have

$$\begin{aligned} 2^{k+1} &> 2k^3 = k^3 + k^3 \geq k^3 + 10k^2 \\ &= k^3 + 3k^2 + 7k^2 \geq k^3 + 3k^2 + 70k \\ &= k^3 + 3k^2 + 3k + 67k \geq k^3 + 3k^2 + 3k + 670 \\ &\leq k^3 + 3k^2 + 3k + 1 = (k+1)^3 \end{aligned}$$

So we have verified $2^{k+1} > (k+1)^3$

□

Exercise 7. *Consider the open sentence* $P(n) : 9 + 13 + \cdots + (4n+5) = \dfrac{4n^2 + 14n + 1}{2}$, *where* $n \in \mathbb{N}$.

(a) *Verify the implication* $P(k) \Rightarrow P(k+1)$ *for an arbitrary positive integer* k.

(b) *Is* $\forall n \in \mathbb{N}, P(n)$ *a true statement?*

Solution.

(a) Assume that

$$9 + 13 + \cdots + (4k+5) = \frac{4k^2 + 14k + 1}{2}.$$

Then

$$9+13+\cdots+(4k+9) = 9+13+\cdots+(4k+5)+(4k+9) = \frac{4k^2+14k+1}{2}+(4k+9) = \frac{4k^2+22k+19}{2}$$

and since

$$4(k+1)^2 + 14(k+1) + 1 = 4k^2 + 8k + 4 + 14k + 14 + 1 = 4k^2 + 22k + 19$$

we have verified the formula for $n = k+1$.

(b) It is a false statement. Observe for $n = 1$, the right hand side is

$$\frac{4(1)^2 + 14(1) + 1}{2} = \frac{19}{2} = 9.5 \neq 9$$

so the equation is not true for $n = 1$, which is our counterexample.

□